$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 5 & 7 & 1 \\ 2 & 1 & 8 & -8 \\ 1 & -4 & -3 & 9 \end{pmatrix}$$

Supponiamo di voler calcolare l'inversa della matrice quadrata A di dimensione (4x4) riportata qui sopra, per prima cosa dobbiamo accertarci che la matrice sia NON singolare ovvero che il suo determinate sia diverso da 0:

det A ≠ 0

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 5 & 7 & 1 \\ 2 & 1 & 8 & -8 \\ 1 & -4 & -3 & 9 \end{pmatrix}$$

Il Teorema di Laplace afferma: "Data una matrice quadrata di ordine n, il suo determinante è uguale alla somma dei prodotti degli elementi di una qualsiasi riga (o colonna) per i rispettivi complementi algebrici".
Il complemento algebrico dell'elemento aij appartenente alla matrice M, è il determinante della matrice che si ottiene cancellando la i-esima riga e la j-esima colonna dalla matrice M, preso con il segno + se i+j è pari, segno – se i+j è dispari.

$det\ A = a_{11} \bullet A_{11} + a_{12} \bullet A_{12} + a_{13} \bullet A_{13} + a_{14} \bullet A_{14}$

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 5 & 7 & 1 \\ 2 & 1 & 8 & -8 \\ 1 & -4 & -3 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 5 & 7 & 1 \\ 2 & 1 & 8 & -8 \\ 1 & -4 & -3 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 5 & 7 & 1 \\ 2 & 1 & 8 & -8 \\ 1 & -4 & -3 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 5 & 7 & 1 \\ 2 & 1 & 8 & -8 \\ 1 & -4 & -3 & 9 \end{pmatrix}$$

$$det\ A = 1 \bullet (-1)^{1+1} \bullet \begin{vmatrix} 5 & 7 & 1 \\ 1 & 8 & -8 \\ -4 & -3 & 9 \end{vmatrix} + \underbrace{0 \bullet (-1)^{1+2} \bullet \begin{vmatrix} -1 & 7 & 1 \\ 2 & 8 & -8 \\ 1 & -3 & 9 \end{vmatrix}} + 3 \bullet (-1)^{1+3} \bullet \begin{vmatrix} -1 & 5 & 1 \\ 2 & 1 & -8 \\ 1 & -4 & 9 \end{vmatrix} + \underbrace{0 \bullet (-1)^{1+4} \bullet \begin{vmatrix} -1 & 5 & 7 \\ 2 & 1 & 8 \\ 1 & -4 & -3 \end{vmatrix}}$$

$$\left|\begin{matrix} 5 & 7 & 1 \\ 1 & 8 & -8 \\ -4 & -3 & 9 \end{matrix}\right| \begin{matrix} 5 & 7 \\ 1 & 8 \\ -4 & -3 \end{matrix}$$

Questi due termini defungono per via dei due 0 davanti, da ciò si intuisce facilmente che conviene scegliere la colonna o la riga con più 0 disponibili così risparmio del tempo nei calcoli!

(5•8•9) + (7•(-8)•(-4)) + (1•1•(-3)) – ((-4)•8•1) – ((-3)•(-8)•5) – (9•1•7) = 360 + 224 – 3 + 32 – 120 – 63 = **430**

$$\left|\begin{matrix} -1 & 5 & 1 \\ 2 & 1 & -8 \\ 1 & -4 & 9 \end{matrix}\right| \begin{matrix} -1 & 5 \\ 2 & 1 \\ 1 & -4 \end{matrix}$$

Minore di ordine 3, posso utilizzare la comoda regola di Sarrus (solo per matrici quadrate 3x3).

((-1)•1•9) + (5•(-8)•1) + (1•2•(-4)) – (1•1•1) – ((-4)•(-8)•(-1)) – (9•2•5) = – 9 – 40 – 8 – 1 + 32 – 90 = **-116**

Adesso possiamo calcolare agevolmente il determinante della matrice A:

det A = 1 • $(-1)^2$ • 430 + 3 • $(-1)^4$ • (-116) = 82

Ora accertato che *det* A ≠ 0 passiamo a calcolare la matrice avente per elementi i complementi algebrici:

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} Cof(a_{1,1}) & Cof(a_{1,2}) & \dots & Cof(a_{1,n}) \\ Cof(a_{2,1}) & Cof(a_{2,2}) & \dots & Cof(a_{2,n}) \\ \vdots & \vdots & . & \vdots \\ Cof(a_{n,1}) & Cof(a_{n,2}) & \dots & Cof(a_{n,n}) \end{bmatrix}^T$$

$$A_{11} = (-1)^{1+1} \bullet \begin{vmatrix} 5 & 7 & 1 \\ 1 & 8 & -8 \\ -4 & -3 & 9 \end{vmatrix} = (-1)^{1+1} \bullet ((5\bullet8\bullet9) + (7\bullet(-8)\bullet(-4)) + (1\bullet1\bullet(-3)) - ((-4)\bullet8\bullet1) - ((-3)\bullet(-8)\bullet5) - (9\bullet1\bullet7)) = 430$$

$$A_{12} = (-1)^{1+2} \bullet \begin{vmatrix} -1 & 7 & 1 \\ 2 & 8 & -8 \\ 1 & -3 & 9 \end{vmatrix} = (-1)^{1+2} \bullet (((-1)\bullet8\bullet9) + (7\bullet(-8)\bullet(1)) + (1\bullet2\bullet(-3)) - (1\bullet8\bullet1) - ((-3)\bullet(-8)\bullet(-1)) - (9\bullet2\bullet7)) = 244$$

$$A_{13} = (-1)^{1+3} \bullet \begin{vmatrix} -1 & 5 & 1 \\ 2 & 1 & -8 \\ 1 & -4 & 9 \end{vmatrix} = (-1)^{1+3} \bullet (((-1)\bullet1\bullet9) + (5\bullet(-8)\bullet(1)) + (1\bullet2\bullet(-4)) - (1\bullet1\bullet1) - ((-4)\bullet(-8)\bullet(-1)) - (9\bullet2\bullet5)) = -116$$

$$A_{14} = (-1)^{1+4} \bullet \begin{vmatrix} -1 & 5 & 7 \\ 2 & 1 & 8 \\ 1 & -4 & -3 \end{vmatrix} = (-1)^{1+4} \bullet (((-1)\bullet1\bullet(-3)) + (5\bullet8\bullet1) + (7\bullet2\bullet(-4)) - (1\bullet1\bullet7) - ((-4)\bullet8\bullet(-1)) - ((-3)\bullet2\bullet5)) = 22$$

$$A_{21} = (-1)^{2+1} \bullet \begin{vmatrix} 0 & 3 & 0 \\ 1 & 8 & -8 \\ -4 & -3 & 9 \end{vmatrix} = (-1)^{2+1} \bullet ((0\bullet8\bullet9) + (3\bullet(-8)\bullet(-4)) + (0\bullet1\bullet(-3)) - ((-4)\bullet8\bullet0) - ((-3)\bullet(-8)\bullet0) - (9\bullet1\bullet3)) = -69$$

$$A_{22} = (-1)^{2+2} \bullet \begin{vmatrix} 1 & 3 & 0 \\ 2 & 8 & -8 \\ 1 & -3 & 9 \end{vmatrix} = (-1)^{2+2} \bullet ((1\bullet8\bullet9) + (3\bullet(-8)\bullet1) + (0\bullet2\bullet(-3)) - ((1)\bullet8\bullet0) - ((-3)\bullet(-8)\bullet1) - (9\bullet2\bullet3)] = -30$$

$$A_{23} = (-1)^{2+3} \bullet \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -8 \\ 1 & -4 & 9 \end{vmatrix} = (-1)^{2+3} \bullet ((1\bullet1\bullet9) + (0\bullet(-8)\bullet1) + (0\bullet2\bullet(-4)) - (1\bullet1\bullet0) - ((-4)\bullet(-8)\bullet1) - (9\bullet2\bullet0)) = 23$$

$$A_{24} = (-1)^{2+4} \bullet \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 8 \\ 1 & -4 & -3 \end{vmatrix} = (-1)^{2+4} \bullet ((1\bullet1\bullet(-3)) + (0\bullet8\bullet1) + (3\bullet2\bullet(-4)) - (1\bullet1\bullet3) - ((-4)\bullet8\bullet1) - ((-3)\bullet2\bullet0)) = 2$$

$$A_{31} = (-1)^{3+1} \bullet \begin{vmatrix} 0 & 3 & 0 \\ 5 & 7 & 1 \\ -4 & -3 & 9 \end{vmatrix} = (-1)^{3+1} \bullet ((0\bullet7\bullet9) + (3\bullet1\bullet(-4)) + (0\bullet5\bullet(-3)) - ((-4)\bullet7\bullet0) - ((-3)\bullet1\bullet0) - (9\bullet5\bullet3)) = -147$$

$$A_{32} = (-1)^{3+2} \bullet \begin{vmatrix} 0 & 3 & 0 \\ 5 & 7 & 1 \\ -4 & -3 & 9 \end{vmatrix} = (-1)^{3+2} \bullet ((1\bullet7\bullet9) + (3\bullet1\bullet1) + (0\bullet(-1)\bullet(-3)) - (1\bullet7\bullet0) - ((-3)\bullet1\bullet1) - (9\bullet(-1)\bullet3)) = -96$$

$$A_{33} = (-1)^{3+3} \bullet \begin{vmatrix} 1 & 0 & 0 \\ -1 & 5 & 1 \\ 1 & -4 & 9 \end{vmatrix} = (-1)^{3+3} \bullet ((1\bullet5\bullet9) + (0\bullet1\bullet1) + (0\bullet(-1)\bullet(-4)) - (1\bullet5\bullet0) - ((-4)\bullet1\bullet1) - (9\bullet(-1)\bullet0)) = 49$$

$$A_{34} = (-1)^{3+4} \bullet \begin{vmatrix} 1 & 0 & 3 \\ -1 & 5 & 7 \\ 1 & -4 & -3 \end{vmatrix} = (-1)^{3+4} \bullet ((1\bullet5\bullet(-3)) + (0\bullet7\bullet1) + (3\bullet(-1)\bullet(-4)) - (1\bullet5\bullet3) - ((-4)\bullet7\bullet1) - ((-3)\bullet(-1)\bullet0)) = -10$$

$$A_{41} = (-1)^{4+1} \bullet \begin{vmatrix} 0 & 3 & 0 \\ 5 & 7 & 1 \\ 1 & 8 & -8 \end{vmatrix} = (-1)^{4+1} \bullet ((0\bullet7\bullet(-8)) + (3\bullet1\bullet1) + (0\bullet5\bullet8) - (1\bullet7\bullet0) - (8\bullet1\bullet0) - ((-8)\bullet5\bullet3)) = -123$$

$$A_{42} = (-1)^{4+2} \bullet \begin{vmatrix} 1 & 3 & 0 \\ -1 & 7 & 1 \\ 2 & 8 & -8 \end{vmatrix} = (-1)^{4+2} \bullet ((1\bullet7\bullet(-8)) + (3\bullet1\bullet2) + (0\bullet(-1)\bullet8) - (2\bullet7\bullet0) - (8\bullet1\bullet1) - ((-8)\bullet(-1)\bullet3)) = -82$$

$$A_{43} = (-1)^{4+3} \bullet \begin{vmatrix} 1 & 0 & 0 \\ -1 & 5 & 1 \\ 2 & 1 & -8 \end{vmatrix} = (-1)^{4+3} \bullet ((1\bullet5\bullet(-8)) + (0\bullet1\bullet2) + (0\bullet(-1)\bullet1) - (2\bullet5\bullet0) - (1\bullet1\bullet1) - ((-8)\bullet(-1)\bullet0)) = 41$$

$$A_{44} = (-1)^{4+4} \bullet \begin{vmatrix} 1 & 0 & 3 \\ -1 & 5 & 7 \\ 2 & 1 & 8 \end{vmatrix} = (-1)^{4+4} \bullet ((1\bullet5\bullet8) + (0\bullet7\bullet2) + (3\bullet(-1)\bullet1) - (2\bullet5\bullet3) - (1\bullet7\bullet1) - (8\bullet(-1)\bullet0)) = 0$$

$$\mathbf{A} = \begin{pmatrix} 430 & 244 & -116 & 22 \\ -69 & -30 & 23 & 2 \\ -147 & -96 & 49 & -10 \\ -123 & -82 & 41 & 0 \end{pmatrix}$$

$$\mathbf{A^t} = \begin{pmatrix} 430 & -69 & -147 & -123 \\ 244 & -30 & -96 & -82 \\ -116 & 23 & 49 & 41 \\ 22 & 2 & -10 & 0 \end{pmatrix}$$

$$\mathbf{A^{-1}} = 1 / 82 \bullet \begin{pmatrix} 430 & -69 & -147 & -123 \\ 244 & -30 & -96 & -82 \\ -116 & 23 & 49 & 41 \\ 22 & 2 & -10 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{5,244} & \mathbf{-0,841} & \mathbf{-1,793} & \mathbf{-1,5} \\ \mathbf{2,976} & \mathbf{-0,366} & \mathbf{-1,171} & \mathbf{-1} \\ \mathbf{-1,415} & \mathbf{0,280} & \mathbf{0,598} & \mathbf{0,5} \\ \mathbf{0,268} & \mathbf{0,024} & \mathbf{-0,122} & \mathbf{0} \end{pmatrix}$$

Facciamo ora la prova del nove sappiamo infatti che:

$$\mathbf{A^{-1} \bullet A = I \; ; \; A \bullet A^{-1} = I}$$

$$\begin{pmatrix} \mathbf{5,244} & \mathbf{-0,841} & \mathbf{-1,793} & \mathbf{-1,5} \\ \mathbf{2,976} & \mathbf{-0,366} & \mathbf{-1,171} & \mathbf{-1} \\ \mathbf{-1,415} & \mathbf{0,280} & \mathbf{0,598} & \mathbf{0,5} \\ \mathbf{0,268} & \mathbf{0,024} & \mathbf{-0,122} & \mathbf{0} \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 5 & 7 & 1 \\ 2 & 1 & 8 & -8 \\ 1 & -4 & -3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$